BRAID MONODROMY FACTORIZATION FOR A NON-PRIME $K3$ SURFACE BRANCH CURVE

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ABSTRACT. In this paper we consider a non-prime $K3$ surface of degree 16, and study a specific degeneration of it, known as the $(2, 2)$-pillow degeneration, [10].

We study also the braid monodromy factorization of the branch curve of the surface with respect to a generic projection onto $\mathbb{CP}^2$.

In [4] we compute the fundamental groups of the complement of the branch curve and of the corresponding Galois cover of the surface.

1. OVERVIEW

Given a projective surface and a generic projection to the plane, the fundamental group of the complement of the branch curve is one of its most important invariants. Our goal is to compute this group and the fundamental group of the Galois cover (which is known to be a certain quotient of the fundamental group of the complement of the branch curve, see [13]). This goal is achieved in [4].

In this paper we deal with a non-prime $K3$ surface of degree 16 which is embedded in $\mathbb{P}^9$. In order to compute the above groups we degenerate the surface into a union of 16 planes. We then project it onto $\mathbb{CP}^2$ to get a degenerated branch curve S_0, from which one can compute the braid monodromy factorization and the branch curve S of the projection of the original $K3$ surface. With this information we will be able to apply the van Kampen Theorem (see [20]) and the regeneration rules (see [18]) to get presentations for the relevant fundamental groups.

The idea of using degenerations for these purposes appears already in [3], [9], [11], [13] and [17]. Degenerations of $K3$ surfaces, of which the one we use here is an example, were constructed in [10] and are called *pillow degenerations*.

The study of braids was begun by Artin in [1] and [2]; see also Chisini [8]. The braid monodromy technique was first presented by Moishezon-Teicher in [12], [14] and [15]. Many examples of computations of braid monodromy have been executed, see for example in [3], [5], [15]. Also in [7] one can find a description of computations of braid monodromy and the fundamental group $\pi_1(\mathbb{CP}^2 - S)$ of the complement of the branch curve of a surface. In this work we encounter new type of singularities (3-points, see Section 4.2.1) which have not been handled before, whose analysis is necessary to give the precise computations of the braid monodromy. Moreover, since the monodromies related to 6-points are quite hard to follow, we present them first in a precise way algebraically, followed then by figures which illustrate the computations.

In [7], the authors also considered the branch curve of a projection of a non-prime $K3$ surface. But they considered quotients of $\pi_1(\mathbb{CP}^2 - S)$ (where S is the branch curve) by a subgroup of commutators (commutators of geometric generators which are mapped to disjoint transpositions by the geometric monodromy representation, see Definition 2.2 in that paper). Their motivation came from the theory of symplectic manifolds and families of projections where the branch curves acquire and may lose pairs of transverse double points with opposite orientations; creating a pair of double points adds a commutation relation. The quotient there is the largest quotient of $\pi_1(\mathbb{CP}^2 - S)$, which is guaranteed to be invariant under these operations, and it is different from the $\pi_1(\mathbb{CP}^2 - S)$ or the group related to the fundamental group of the Galois cover, which is our aim (see [4]).

The paper is divided as follows. In Section 2 we give the definition of a degeneration and explain briefly the computations from [10]. In Section 3 we review the general setup and the notion of the braid monodromy. In Section 4 we recover the relevant properties of the branch curve S of the generic projection to the plane of the $K3$ surface by using regeneration techniques. Section 5 states the braid monodromy factorization of Δ_{48}^2 of S and its related invariance properties. We explain why the computations of Δ_{48}^2 and its invariance are necessary for this work and for future work.

2. $K3$ surfaces and their degenerations

2.1. $K3$ **surfaces.** Compact complex surfaces are classified into four broad categories, based on the growth rate of sections of powers of the canonical class. Such sections can either be always zero (rational or ruled surfaces); form vector spaces of bounded dimension, or have spaces of sections whose dimensions grow linearly (elliptic surfaces) or quadratically (surfaces of general type) with the power. For surfaces for which these sections are bounded, some multiple of the canonical bundle is trivial, and there are nine separate families up to complex deformation. The surfaces of this type which are simply connected in fact have

trivial canonical bundle, and are called $K3$ surfaces; the invariants for such surfaces are $p_g = 1$, $q = 0$, $e = 24$, and $h^{1,1} = 22$, see [9] and [10]. The most common example of a $K3$ surface is a smooth quartic surface in $\mathbb{P}^3$. The moduli space of all $K3$ surfaces is 20-dimensional.

Most $K3$ surfaces are not algebraic; the algebraic ones are classified by an infinite collection (depending on an integer $g \geq 2$) of 19-dimensional moduli spaces. The general member of the family has a rank one Picard group, generated by an ample class H with $H^2 = 2g - 2$; the general member of the linear system $|H|$ is a smooth curve of genus g, and this linear system maps the $K3$ surface to $\mathbb{P}^g$ as a surface of degree $2g - 2$. For example, the quartic surfaces in $\mathbb{P}^3$ form the family with $g = 3$. The integer g is called the *genus* of the family.

The $K3$ surfaces may also be embedded by a multiple cH of the primitive class H; this will exhibit the family of $K3$ surfaces of genus g as surfaces of degree $D = c^2(2g - 2)$ in $\mathbb{P}^G$, where $G = 1 + c^2(g - 1)$, whose hyperplane section is an element of $|cH|$, and therefore is a curve of genus G.

2.2. Degenerations of $K3$ surfaces.

Let us start this section with recalling the definition of degeneration from [17].

Definition 1. Projective degeneration *Let Δ be the unit disc, and X, Y be algebraic surfaces (or more generally algebraic varieties). Suppose that $k : Y \to \mathbb{CP}^n$ and $k' : X \to \mathbb{CP}^n$ are projective embeddings. We say that k' is a* **projective degeneration** *of k if there exist a flat family $\pi : V \to \Delta$, and an embedding $F : V \to \Delta \times \mathbb{CP}^n$, such that F composed with the first projection is π, and:*

(a) *$\pi^{-1}(0) \simeq X$;*
(b) *there is a $t_0 \neq 0$ in Δ such that $\pi^{-1}(t_0) \simeq Y$;*
(c) *the family $V - \pi^{-1}(0) \to \Delta - 0$ is smooth;*
(d) *restricted to $\pi^{-1}(0)$, $F = 0 \times k'$ under the identification of $\pi^{-1}(0)$ with X;*
(e) *restricted to $\pi^{-1}(t_0)$, $F = t_0 \times k$ under the identification of $\pi^{-1}(t_0)$ with Y.*

In [10], the authors constructed a degeneration of embedded $K3$ surfaces, to a union of D planes, meeting according to the combinatorics of a certain triangulation of the 2-sphere. Such degenerations are called *Type III* degenerations, and there are many such available. The particular one constructed there was formed by first decomposing the sphere into two rectangles, then decomposing each rectangle into ab squares (by decomposing the sides of the rectangles into a and b line segments), and finally decomposing each square into two triangles (planes). The parameters a and b are free to choose, and the construction exhibits a degeneration of $K3$ surfaces into $4ab$ planes. If c is the greatest common divisor of a and

b, the general member of the family is a smooth $K3$ surface embedded by the multiple c of the generator H of the Picard group. Hence using the above notation, $4ab = 2c^2(g-1)$, so that $g = 1 + 2(ab/c^2)$ and $G = 1 + 2ab$ (the number of coordinate points in $\mathbb{P}^G$ is $G + 1$). The boundary of the two rectangular arrays of planes contains $2a + 2b$ lines, where the identification is taking place.

This degeneration of a $K3$ surface, is called a **pillow degeneration**, see [10].

This article proceeds by making further analyses with the $a = b = 2$ case. Given a non-prime $K3$ surface of degree 16, we get the pillow degeneration $(K3)_0$, as depicted in Figure 1. We quote the main result from [10].

Theorem 2. *The $(2,2)$-pillow degeneration $(K3)_0$ is a total degeneration of a smooth $K3$ surface of degree 16 in $\mathbb{P}^9$. The smooth $K3$ surface is a re-embedding of a quartic surface in $\mathbb{P}^3$ via the linear system of quadrics.*
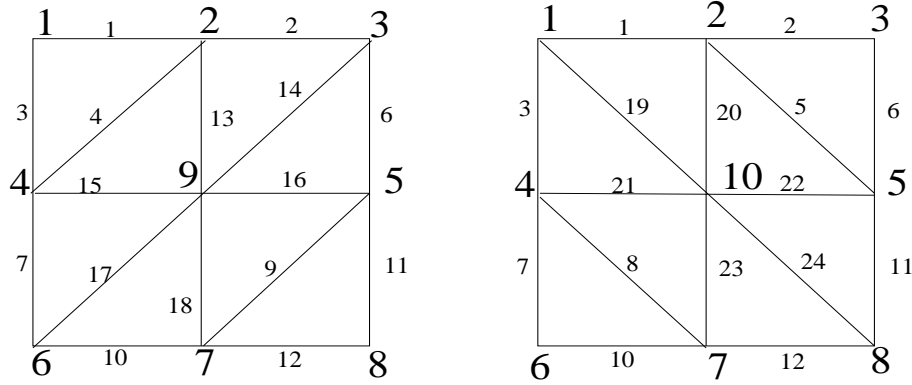

FIGURE 1. The $(2,2)$-pillow degeneration

For the regeneration process we have to fix a numbering of the vertices and lines. The boundary points are labeled from 1 to 8, while the interior point on the top is 9 and on the bottom is 10. The 16 planes meet each other along a total of 24 lines, each joining two of the 10 coordinate points. The numbering of the vertices induces a lexicographic numbering of the lines as follows. If L has endpoints $a < b$ and M has endpoints $c < d$ then $L < M$ if $b < d$ or $d = b$ and $a < c$. This gives a total ordering of the lines, which we interpret as a numbering from 1 to 24, as shown in Figure 1.

Note that the coordinate points 1, 3, 6, and 8 at the corners are each contained in three distinct planes, while all other points are each contained in six planes. We call these two types 3-points and 6-points respectively.

A general projection $f_0 : (K3)_0 \to \mathbb{CP}^2$ will also be a degeneration of a general such projection of the smooth $K3$ surface. Under f_0, each of the 16 planes is mapped isomorphically to $\mathbb{CP}^2$. The ramification locus R_0 of f_0 is the closed subset of $(K3)_0$, where f_0 is not a local

isomorphism. Here R_0 is exactly the 24 lines. Let $S_0 = f_0(R_0)$ be the degenerated branch curve; it is a line arrangement, composed of the images of the 24 lines (each counted twice and each of which is a line in the plane).

3. THE BRAID MONODROMY NOTION

Consider the following setting (Figure 2). S is an algebraic curve in $\mathbb{C}^2$, with $p = \deg(S)$. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a generic projection onto the first coordinate. Define the fiber $K(x) = \{y \mid (x, y) \in S\}$ in S over a fixed point x, projected to the y-axis. Define $N = \{x \mid \#K(x) < p\}$ and $M' = \{s \in S \mid \pi_{|s}$ is not étale at $s\}$; note that $\pi(M') = N$. Let $\{A_j\}_{j=1}^q$ be the set of points of M' and $N = \{x_j\}_{j=1}^q$ their projection on the x-axis. Recall that π is generic, so we assume that $\#(\pi^{-1}(x) \cap M') = 1$ for every $x \in N$. Let E (resp. D) be a closed disk on the x-axis (resp. the y-axis), such that $M' \subset E \times D$ and $N \subset \mathrm{Int}(E)$. We choose $u \in \partial E$ a real point far enough from the set N, so that $x << u$ for every $x \in N$. Define $\mathbb{C}_u = \pi^{-1}(u)$ and number the points of $K = \mathbb{C}_u \cap S$ as $\{1, \ldots, p\}$.
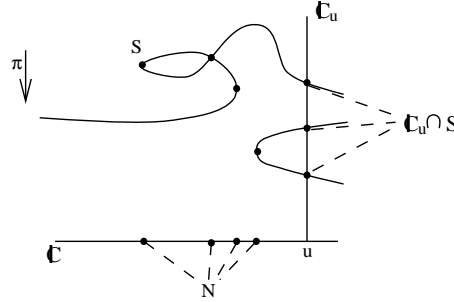

FIGURE 2. General setting

We now construct a g-base for the fundamental group $\pi_1(E - N, u)$. Take a set of paths $\{\gamma_j\}_{j=1}^q$ which connect u with the points $\{x_j\}_{j=1}^q$ of N. Now encircle each x_j with a small oriented counterclockwise circle c_j. Denote the path segment from u to the boundary of this circle as γ'_j. We define an element (a loop) in the g-base as $\delta_j = \gamma'_j c_j \gamma'^{-1}_j$. Let $B_p[D, K]$ be the braid group, and let $H_1, \ldots, H_{p-1}$ be its frame (for complete definitions, see [14, Section III.2]). The braid monodromy of S [2] is a map $\varphi : \pi_1(E - N, u) \to B_p[D, K]$ defined as follows: every loop in $E - N$ starting at u has liftings to a system of p paths in $(E - N) \times D$ starting at each point of $K = 1, \ldots, p$. Projecting them to D we obtain p paths in D defining a motion $\{1(t), \ldots, p(t)\}$ (for $0 \le t \le 1$) of p points in D starting and ending at K. This motion defines a braid in $B_p[D, K]$. By the Artin Theorem [15], for $j = 1, \ldots, q$, there exists a halftwist $Z_j \in B_p[D, K]$ and $\epsilon_j \in \mathbb{Z}$, such that $\varphi(\delta_j) = Z_j^{\epsilon_j}$, where Z_j is a halftwist and

$\epsilon_j = 1, 2$ or 3 (for an ordinary branch point, a node, or a cusp respectively). We now explain how to describe this Z_j.

First we recall a definition of an almost real curve from [15], as we have such curves here.

Definition 3. *A curve S is called an* almost real curve *if*

(1) $N \subseteq E \cap \mathbb{R}$,

(2) $N \subset E - \partial E$,

(3) $\forall x \in E \cap \mathbb{R} - N, \#(K \cap \mathbb{R})(x) \geq p - 1$,

(4) $\forall x \in N, \#\pi^{-1}(x) \cap M' = 1$,

(5) *The singularities can be*

    (a) *a branch point, topologically locally equivalent to $y^2 + x = 0$ or $y^2 - x = 0$,*

    (b) *a tacnode (a line is tangent to a conic)*

    (c) *an intersection of m smooth branches, transversal to each other.*

We explain how to get the braid monodromy around each singularity in S. Let A_j be a singularity in S and denote by x_j its projection by π to the x-axis. We choose a point x'_j next to x_j, such that $\pi^{-1}(x'_j)$ is a typical fiber. If A_j is (b), (c) or (d), then x'_j is on the right side of x_j. If A_j is (a), then x'_j is on the left side of x_j (the typical fiber in case (a), which is on the left side of this singularity, intersects the conic in two real points). We encircle A_j with a very small circle in such a way that the typical fiber $\pi^{-1}(x'_j)$ intersects the circle in two points, say a, b. We fix a skeleton $\xi_{x'_j}$ which connects a and b, and denote it as $< a, b >$. The Lefschetz diffeomorphism Ψ ([3, Subsection 1.9.5]) allows us to get a resulting skeleton $(\xi_{x'_j})\Psi$ in the typical fiber $\mathbb{C}_u$. This one defines a motion of its two endpoints. This motion induces a halftwist $Z_j = \Delta < (\xi_{x'_j})\Psi >$. As above, $\varphi(\delta_j) = \Delta < (\xi_{x'_j})\Psi >^{\epsilon_j}$. The braid monodromy factorization associated to S is

$$\Delta_p^2 = \prod_{j=1}^{q} \varphi(\delta_j).$$

4. The branch curve S_0 and its regeneration

4.1. **The branch curve S_0.** The degenerated object $(K3)_0$ has $\{j\}_{j=1}^{10}$ as vertices. By the projection $f_0 : K3_0 \to \mathbb{CP}^2$, we obtain a line arrangement $S_0 = \bigcup_{i=1}^{24} L_i$, and the projections $f_0(j) = j$ are singularities of S_0. The 3-points are $1, 3, 6, 8$ and the 6-points are $2, 4, 5, 7, 9, 10$. In the following subsection we regenerate in neighborhoods of these singularities. The monodromies will be given in the next section.

Besides these singularities in S_0, there are also *parasitic intersections*: these come from lines in $(K3)_0$ which do not intersect, but when projecting $(K3)_0$ onto $\mathbb{CP}^2$, they will intersect. Denote each line in $(K3)_0$ as a pair (by its two end vertices). Let u be a point, $u \notin S_0$,

such that $\#\mathbb{C}_u = 24$, and $q_i = i \cap \mathbb{C}_u$ a real point. Take two non-intersecting lines $p = (i, k)$ and $t = (j, \ell)$ in $(K3)_0$.

Notation 4. *We denote by Z_{ij} (resp. $\bar{Z}_{ij}$) the counterclockwise halftwist of i and j along a path below (resp. above) the real axis. If P is a set of points between i and j, $\overset{(P)}{Z_{ij}}$ denotes the path from i to j going above the points in P and below the points not in P. Conjugation of braids is defined as $a^b = b^{-1}ab$.*

We first compute the products $D_t := \prod\limits_{\substack{p<t \\ p\cap t=\emptyset}} \tilde{Z}_{pt}^2$ as explained in [14, Theorem IX.2.1].

$$D_1 = D_2 = D_4 = Id \ , \ \ D_3 = Z_{2\,3}^2 \ , \ \ D_5 = \bar{Z}_{3\,5}^2 \ , \ \ D_6 = \prod_{i=1,3,4} \overset{(5)}{\bar{Z}_{i\,6}^2} \ , \ \ D_7 = \prod_{i=1,2,5,6} \bar{Z}_{i\,7}^2$$

$$D_8 = \prod_{i=1,2,5,6} \bar{Z}_{i\,8}^2 \ , \ \ D_9 = \prod_{i=1-4,7} \overset{(8)}{\bar{Z}_{i\,9}^2} \ , \ \ D_{10} = \prod_{i=1}^{6} \overset{(8)(9)}{\bar{Z}_{i\,10}^2} \ , \ \ D_{11} = \prod_{i=1-4,7,8,10} \bar{Z}_{i\,11}^2 \ ,$$

$$D_{12} = \prod_{i=1}^{7} \overset{(11)}{\bar{Z}_{i\,12}^2} \ , \ \ D_{13} = \prod_{\substack{i=3 \\ i\neq 4,5}}^{12} \bar{Z}_{i\,13}^2 \ , \ \ D_{14} = \prod_{\substack{i=1 \\ i\neq 2,6}}^{12} \overset{(13)}{\bar{Z}_{i\,14}^2} \ , \ \ D_{15} = \prod_{\substack{i=1 \\ i\neq 3,4,7,8}}^{12} \overset{(13)(14)}{\bar{Z}_{i\,15}^2} \ ,$$

$$D_{16} = \prod_{\substack{i=1 \\ i\neq 5,6,9,11}}^{12} \overset{(13)-(15)}{\bar{Z}_{i\,16}^2} \ , \ \ D_{17} = \prod_{\substack{i=1 \\ i\neq 7,10}}^{12} \overset{(13)-(16)}{\bar{Z}_{i\,17}^2} \ , \ \ D_{18} = \prod_{\substack{i=1 \\ i\neq 8-10}}^{11} \overset{(13)-(17)}{\bar{Z}_{i\,18}^2} \ , \ \ D_{19} = \prod_{\substack{i=2 \\ i\neq 3}}^{18} \bar{Z}_{i\,19}^2 \ ,$$

$$D_{20} = \prod_{\substack{i=3 \\ i\neq 4,5,13}}^{18} \overset{(19)}{\bar{Z}_{i\,20}^2} \ , \ \ D_{21} = \prod_{\substack{i=1 \\ i\neq 3,4,7,8,15}}^{18} \overset{(19)(20)}{\bar{Z}_{i\,21}^2} \ , \ \ D_{22} = \prod_{\substack{i=1 \\ i\neq 5,6,9,11,16}}^{18} \overset{(19)-(21)}{\bar{Z}_{i\,22}^2} \ ,$$

$$D_{23} = \prod_{\substack{i=1 \\ i\neq 8-10,12}}^{17} \overset{(19)-(22)}{\bar{Z}_{i\,23}^2} \ , \ \ D_{24} = \prod_{\substack{i=1 \\ i\neq 11,12}}^{18} \overset{(19)-(23)}{\bar{Z}_{i\,24}^2} \ .$$

Define the parasitic intersection braids as

(1) $$\tilde{C}_j = \prod_{j\in t} D_t,$$

where j is the <u>smallest</u> endpoint in the line t. In our case, $\tilde{C}_1 = D_1 \cdot D_3 \cdot D_{19}$, $\tilde{C}_2 = D_2 \cdot D_4 \cdot D_5 \cdot D_{13} \cdot D_{20}$, $\tilde{C}_3 = D_6 \cdot D_{14}$, $\tilde{C}_4 = D_7 \cdot D_8 \cdot D_{15} \cdot D_{21}$, $\tilde{C}_5 = D_9 \cdot D_{11} \cdot D_{16} \cdot D_{22}$, $\tilde{C}_6 = D_{10} \cdot D_{17}$, $\tilde{C}_7 = D_{12} \cdot D_{18} \cdot D_{23}$, $\tilde{C}_8 = D_{24}$, $\tilde{C}_9 = \tilde{C}_{10} = Id$.

4.2. **The regeneration of S_0.** The degenerated branch curve S_0 of $(K3)_0$ has degree 24. However each of the 24 lines of S_0 should be counted as a double line in the scheme-theoretic branch locus, since it arises from a line of nodes. Another way to see this is to note that the

regeneration of $(K3)_0$ induces a regeneration of S_0 in such a way that each point, say c, on the typical fiber is replaced by two nearby points c, c'.

The curve S_0 has 3-points, 6-points and parasitic intersections. In the forthcoming subsections we explain how to regenerate the curve in neighbourhoods of these points. The resulting branch curve S will have degree 48.

4.2.1. *Regeneration of* 3-*points.* The 3-points in S_0 are $1, 3, 6, 8$, see Figure 1. The regeneration is divided into steps. We explain each step in two levels, first dealing with the surface and then with the branch curve.

In the surface level, each diagonal is replaced with a conic by a partial regeneration. Focusing on a 3-point, we have a partial regeneration of two of the planes to a quadric surface. We get one quadric and one plane, which is tangent to the quadric. The plane and the quadric meet along two lines (one from each ruling of the quadric).

For the regeneration of the branch curve, we need the following lemma from [18].

Lemma 5. *Let V be a projective algebraic surface, and D' be a curve in V. Let $f : V \to \mathbb{CP}^2$ be a generic projection. Let $S \subseteq \mathbb{CP}^2, S' \subset V$ be the branch curve of f and the corresponding ramification curve. Assume S' intersects D' at a point α'. Let $D = f(D')$ and $\alpha = f(\alpha')$. Assume that there exist neighbourhoods of α and α', such that $f_{|S'}$ and $f_{|D'}$ are isomorphisms. Then D is tangent to S at α.*

At the branch curve level, we have two double lines (coming from the intersection of the plane and the quadric) and one conic (coming from the branching of the quadric over the plane). According to the above lemma, the conic is tangent to each of the two double lines.

As far as the branch points go, one of the two branch points of the conic is far away from the 3-point, and the other one is close to the 3-point; see for example Figure 3.
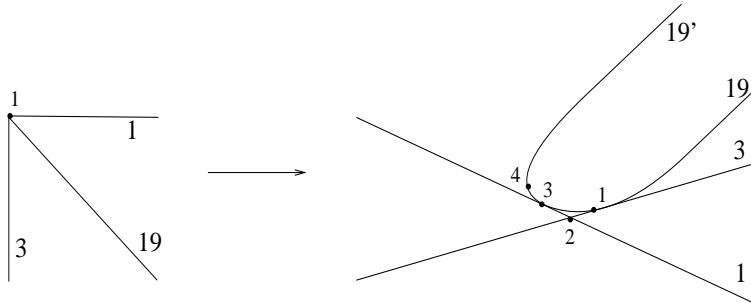

FIGURE 3. Partial regeneration in a neighbourhood of the 3-point 1

In the next step of the regeneration, at the curve level, we use regeneration Lemmas from [15]. The two tangent points regenerate to three cusps each (giving a total of six) and the

intersection point of the two double lines gives eight more branch points. One can think of this as first giving four nodes, then each node giving two branch points.

At the surface level it means that we get a smooth surface which locally looks like a cubic in $\mathbb{CP}^3$ (degenerating to a triple of planes).

4.2.2. *Regeneration of* 6-*points.* The 6-points in S_0 are $2, 4, 5, 7, 9, 10$. Each of these points is the projection to the plane of a cone over a cycle of independent lines spanning a $\mathbb{CP}^5$. Hence, locally, the surface lives in $\mathbb{CP}^6$ and consists of six planes through a point.

Let us understand how the regeneration of a 6-point occurs, by focusing for example on a neighbourhood around 2 in S_0. The first regeneration consists in smoothing out the lines 4 and 5 and therefore replacing four of these pairwise adjacent but opposite planes with two quadrics, see Figure 4.
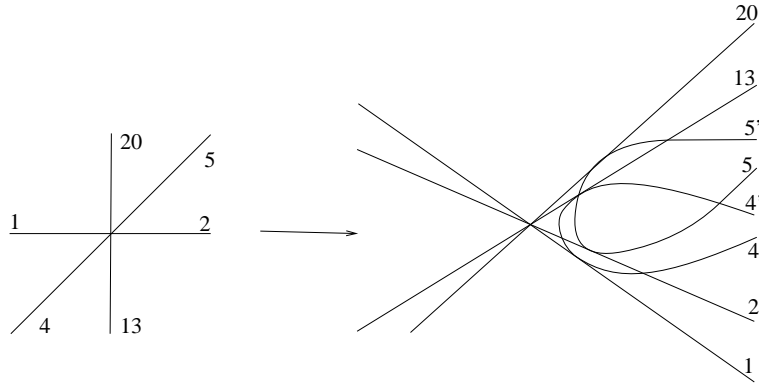
FIGURE 4. Partial regeneration in a neighbourhood of the 6-point 2

Then one smooths out the lines 13 and 20. This produces two cubic rational normal scrolls meeting along the lines 1 and 2. Now the union of these lines on each scroll is homologous to a non singular conic, so we can further regenerate to make the two scrolls meet along a smooth conic (in this way 1 and 2 are replaced by a smooth double conic). We visualize a purely local figure (Figure 5), in order to understand this step.

Finally one smooths the conic too and arrives to a Del-Pezzo sextic in $\mathbb{CP}^6$.

A similar regeneration occurs in the neighbourhoods of the other 6-points.
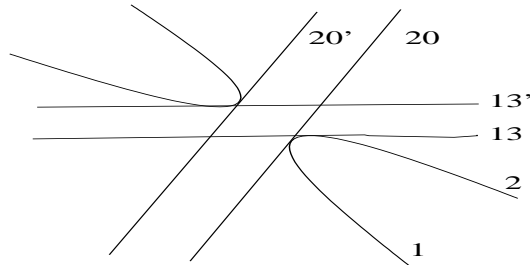

FIGURE 5. Second step of regeneration

5. THE BRAID MONODROMY FACTORIZATION Δ_{48}^2

In Section 4 we explained how to obtain the branch curve S, which has degree 48. At this point we will compute the braid monodromy factorization Δ_{48}^2 of S.

In Subsection 5.1 we formulate braids and regeneration rules. In Subsection 5.2 we compute Δ_{48}^2. In Subsections 5.3 and 5.4 we show an invariance property of Δ_{48}^2, and emphasise its importance.

5.1. **The regeneration rules.** Recall that Z_{ij} is a counterclockwise halftwist of two points i and j. We start with an example that illustrates conjugated braids.

Example 6. *The path on the left-hand side in Figure 6 is constructed as follows: take a path z_{34} and conjugate it by the fulltwist $\bar{Z}_{13}^2$ (1 encircles 3 counterclockwise while moving above the axis). We get the left-hand side path $z_{34}^{\bar{Z}_{13}^2}$. The right-hand side path is constructed as follows: take again z_{34} and conjugate it first by Z_{23}^2 (3 encircles 2 counterclockwise) and then by Z_{13}^2 (3 encircles 1 counterclockwise while moving below the axis). We get the right-hand side path $z_{34}^{Z_{23}^2 Z_{13}^2}$. The related halftwists of $z_{34}^{\bar{Z}_{13}^2}$ and $z_{34}^{Z_{23}^2 Z_{13}^2}$ are $Z_{34}^{\bar{Z}_{13}^2}$ and $Z_{34}^{Z_{23}^2 Z_{13}^2}$ respectively.*

FIGURE 6. Example of conjugated braids

In Section 4.2 we explained that by the regeneration one gets the original branch curve S. Let N, M, u be as in Section 3. The set $K = S \cap \mathbb{C}_u$ are the intersection points of the curve S with the typical fiber $\mathbb{C}_u$; $\#K = 48$. Recall that by the regeneration, each point c in $K_0 = S_0 \cap \mathbb{C}_u$ is replaced by two close points in K, say c, c'.

We define the braid Z_{ij}^2 to be a fulltwist of j around i, and $Z_{i'j}^2$ to be a fulltwist of j around i'. The braid $Z_{ii',j}^2$ is obtained by a regeneration (the point i on the typical fiber is replaced by i, i') and it is a fulltwist of j around i and i'. This braid (and similar ones) is formulated in the following lemma:

Lemma 7. [3, Lemma 2.5]
The following formulas hold: $Z_{ii',j}^2 = Z_{i'j}^2 Z_{ij}^2$, $Z_{i',jj'}^2 = Z_{i'j'}^2 Z_{i'j}^2$, $Z_{i',jj'}^{-2} = Z_{i'j}^{-2} Z_{i'j'}^{-2}$, $\bar{Z}_{i',jj'}^{-2} = \bar{Z}_{i'j'}^{-2} \bar{Z}_{i'j}^{-2}$, $Z_{i',j}^{-2} = Z_{ij}^{-2} Z_{i'j}^{-2}$, $Z_{ii',jj'}^2 = Z_{i',jj'}^2 Z_{i,jj'}^2$, $Z_{ii',jj'}^{-2} = Z_{i,jj'}^{-2} Z_{i',jj'}^{-2}$.

In the following regeneration rules we shall describe what happens to a factor in a factorized expression of a braid monodromy by a regeneration.

Theorem 8. First regeneration rule [18, p. 336]
A factor of the form Z_{ij} regenerates to $Z_{ij'} Z_{i'j}$.

Theorem 9. Second regeneration rule [18, p. 337]

A factor of the form Z_{ij}^2 regenerates to $Z_{ii',j}^2$, $Z_{i,jj'}^2$ or $Z_{ii',jj'}^2$.

Theorem 10. Third regeneration rule [18, p. 337]

A factor of the form Z_{ij}^4 regenerates to $Z_{i,jj'}^3 = (Z_{ij}^3)^{Z_{jj'}} \cdot (Z_{ij}^3) \cdot (Z_{ij}^3)^{Z_{jj'}^{-1}}$ or to $Z_{ii',j}^3 = (Z_{ij}^3)^{Z_{ii'}} \cdot (Z_{ij}^3) \cdot (Z_{ij}^3)^{Z_{ii'}^{-1}}.$

5.2. **The factorization.** The braid monodromy factorization Δ_{48}^2 of S is $\prod_{i=1}^{10} C_i \varphi_i$, where C_i are the regenerations of the parasitic intersection braids from Section 4.1, and φ_i are the local braid monodromies which we get when regenerating around the singularities $1, \ldots, 10$.

5.2.1. *Braid monodromies related to 3-points.* The 3-points are $1, 3, 6, 8$. We concentrate in the neighbourhood of 1, see Figure 3. First the diagonal line 19 regenerates to a conic Q_{19} which is tangent to the two other lines 1 and 3, see Lemma 5. We compute the braid monodromy φ_1 of the resulting curve.

Proposition 11. *The local braid monodromy φ_1 is*

$$\varphi_1 = Z_{3\ 3',19}^3 \cdot (Z_{1\ 3} \cdot Z_{1\ 3} \cdot Z_{1'\ 3} \cdot Z_{1'\ 3} \cdot Z_{1\ 3'} \cdot Z_{1\ 3'} \cdot Z_{1'\ 3'} \cdot Z_{1'\ 3'})^{Z_{3\ 3',19}^2} \cdot$$

$$Z_{1\ 1',19}^3 \cdot Z_{19\ 19'}^{Z_{3\ 3',19}^2 Z_{1\ 1',19}^2}.$$

Proof. We follow Figure 3. Let $\pi_1 : E \times D \to E$ be the projection to E.

Let $\{j\}_{j=1}^4$ be singular points of π_1 as follows: 1, 3 are the tangent points of Q_{19} with the lines L_3, L_1 respectively, 2 is the intersection point of the lines L_1, L_3, and 4 is the branch point in Q_{19}.

Let $N = \{x(j) = x_j \mid 1 \leq j \leq 4\}$, such that $N \subset E - \partial E, N \subset E$. Take $u \in \partial E$, such that $\mathbb{C}_u$ is a typical fiber and $M \in \mathbb{C}_u$ is a real point. Recall that $K = K(M)$. $K = \{1, 3, 19, 19'\}$, such that the points are real and $1 < 3 < 19 < 19'$. Let $i = L_i \cap K$ for $i = 1, 3$ and $\{19, 19'\} = Q_{19} \cap K$.

We are looking for $\varphi_M(\delta_j)$ for $j = 1, \ldots, 4$. So we choose a g-base $\{\delta_j\}_{j=1}^4$ of $\pi_1(E - N, u)$, such that each δ_j is constructed from a path γ_j below the real line and a counterclockwise small circle around the points in N.

The diffeomorphism which is induced from passing through a branch point was defined in [3] and [15] by $\Delta_{I_2\mathbb{R}}^{\frac{1}{2}} < k >$. We recall the precise definition from [3]. Consider a typical fiber on the left side of a branch point (locally defined by $y^2 - x = 0$). The typical fiber intersects the conic in two complex points. Passing through this point, these points become real on the right-hand side typical fiber. The two points move to the k'th place and rotate in a counterclockwise 90° twist. They become real and numbered as $k, k+1$.

First find the skeleton $\xi_{x'_j}$ related to each singular point, as explained in Section 3. Then compute the local diffeomorphisms δ_j induced from singular points j. Since the points 1 and 3 (resp. 2) are tangent points (resp. a node), the diffeomorphisms δ_1 and δ_3 are each of degree 2 (resp. 1).

j	$\xi_{x'_j}$	ϵ_j	δ_j
1	$<3,19>$	4	$\Delta^2 <3,19>$
2	$<1,3>$	2	$\Delta <1,3>$
3	$<3,19>$	4	$\Delta^2 <3,19>$
4	$<19,19'>$	1	$\Delta^{\frac{1}{2}}_{I_2\mathbb{R}} <19>$

Using [3, Theorems 1.41, 1.44] and [15], we compute the skeleton $(\xi_{x'_j})\Psi_{\gamma'_j}$ to each j by applying to the skeleton $\xi_{x'_j}$ the product $\prod\limits_{i=j-1}^{1} \delta_i$.

$$(\xi_{x'_1})\Psi_{\gamma'_1} = <3,19> = z_{3\ 19}$$
$$\varphi_M(\delta_1) = Z^4_{3\ 19}$$

$$(\xi_{x'_2})\Psi_{\gamma'_2} = <1,3>\Delta^2 <3,19> = z_{1\ 3}^{Z^2_{3\ 19}}$$
$$\varphi_M(\delta_2) = Z^2_{1\ 3}{}^{Z^2_{3\ 19}}$$

$$(\xi_{x'_3})\Psi_{\gamma'_3} = <3,19>\Delta <1,3>\Delta^2 <3,19> = z_{1\ 19}$$
$$\varphi_M(\delta_3) = Z^4_{1\ 19}$$
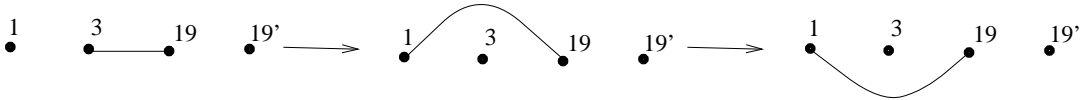
$$(\xi_{x'_4})\Psi_{\gamma'_4} = <19,19'>\Delta^2 <3,19>\Delta <1,3>\Delta^2 <3,19> = z_{19\ 19'}^{Z^2_{3\ 19}Z^2_{1\ 19}}$$
$$\varphi_M(\delta_4) = Z_{19\ 19'}{}^{Z^2_{3\ 19}Z^2_{1\ 19}}$$
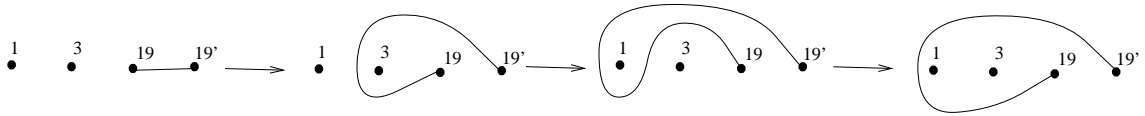
Now, by Theorem 10, in the regeneration process each one of the tangent points regenerates to three cusps. Therefore, the factors $Z^4_{3\ 19}$ and $Z^4_{1\ 19}$ regenerate to $Z^3_{3\ 3',19}$ and $Z^3_{1\ 1',19}$ respectively. As explained in Subsection 4.2.1, the node is replaced by eight branch points: since the lines 1 and 3 double to be 1, 1' and 3, 3' respectively, we get four nodes, the

intersections of 1 and 3, $1'$ and 3, 1 and $3'$, $1'$ and $3'$. Each node regenerates to two branch points. For 1 and 3 we get easily the braid monodromy $Z_{1\ 3} \cdot Z_{1\ 3}$. For the other ones we get $Z_{1'\ 3} \cdot Z_{1'\ 3}$, $Z_{1\ 3'} \cdot Z_{1\ 3'}$ and $Z_{1'\ 3'} \cdot Z_{1'\ 3'}$. In Figure 7 we show the paths which are related to the braids in φ_1.
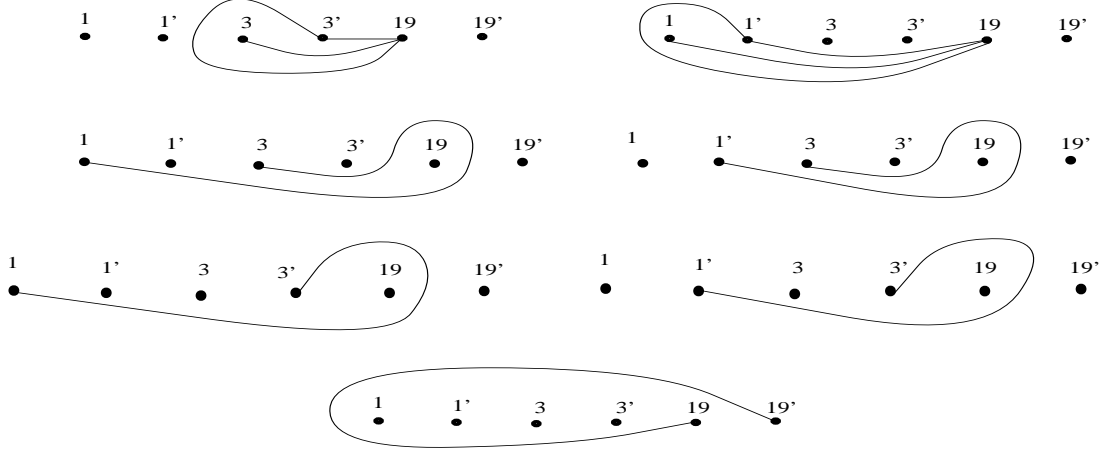

FIGURE 7.

The computations for the points $3, 6$ and 8 are the same. One has to exchange the indices $1, 3$ and 19 to $2, 6$ and 14 (for the point 3), to $7, 10$ and 17 (for the point 6) and to $11, 12$ and 24 (for the point 8) respectively. Therefore we can formulate their related monodromies.

Theorem 12. *The local braid monodromy around the 3-point $m = 1, 3, 6$ or 8 is formulated as follows:*

$$\varphi_m = \quad Z_{j\ j',k}^3 \cdot (Z_{i\ j} \cdot Z_{i\ j} \cdot Z_{i'\ j} \cdot Z_{i'\ j} \cdot Z_{i\ j'} \cdot Z_{i\ j'} \cdot Z_{i'\ j'} \cdot Z_{i'\ j'})^{Z_{j\ j',k}^2} \cdot Z_{i\ i',k}^3 \cdot Z_{k\ k'}^{Z_{j\ j',k}^2 Z_{i\ i',k}^2},$$

where i, j and k are the three lines which meet at the 3-point m and satisfy $i < j < k$.

5.2.2. *Braid monodromies related to 6-points.* The 6-points in S_0 are $2, 4, 5, 7, 9, 10$. In Section 4.2.2 we recalled the regeneration near the 6-point 2. The regenerations near $4, 5, 7, 9, 10$ are similar to the one around 2, but they differ in the indices.

Regenerations in the neighbourhood of a 6-points were studied carefully in [3], [5], [6] and [18]. Therefore, we only state the resulting monodromies.

Theorem 13. *The local braid monodromy φ_2 has the following form*

$$\varphi_2 = Z^2_{2\,2',4} \cdot Z^2_{4',5\,5'} \cdot Z^3_{1\,1',4} \cdot Z^3_{4',13\,13'}{}^{Z^2_{4',5\,5'}} \cdot Z^2_{2\,2',4}{}^{Z^2_{1\,1',4}} \cdot Z^2_{4',5\,5'}{}^{Z^{-2}_{5\,5',13\,13'}} \cdot$$

$$Z_{4\,4'}{}^{Z^2_{4',13\,13'}Z^2_{4',5\,5'}Z^2_{2\,2',4}Z^2_{1\,1',4}} \cdot Z^2_{4',20\,20'}{}^{Z^2_{4',13\,13'}Z^2_{4',5\,5'}} \cdot$$

$$Z^2_{4,20\,20'}{}^{Z^2_{4,13\,13'}Z^2_{4,5\,5'}Z^2_{4\,4'}Z^2_{2\,2',4}Z^2_{1\,1',4}} \cdot \left(G \cdot \left(F \cdot F^{Z^{-1}_{20\,20'}Z^{-1}_{13\,13'}} \right)^{Z^2_{2\,2',5}Z^2_{1\,1',5}} \right)^{Z^2_{2\,2',4}Z^2_{1\,1',4}},$$

where

$$G = Z^2_{5',13\,13'} \cdot Z^3_{2\,2',5} \cdot Z^3_{5',20\,20'}{}^{Z^2_{5',13\,13'}} \cdot Z^2_{5',13\,13'}{}^{Z^{-2}_{13\,13',20\,20'}} \cdot Z_{5\,5'}{}^{Z^2_{2\,2',5}Z^2_{5',20\,20'}Z^2_{5',13\,13'}} \cdot$$

$$Z^2_{1\,1',5} \cdot Z^2_{1\,1',5'}{}^{Z^2_{5',20\,20'}Z^2_{5',13\,13'}}$$

and

$$F = Z^3_{2\,2',13} \cdot Z^2_{13'\,20} \cdot Z^2_{13\,20}{}^{Z^2_{2\,2',13}} \cdot Z^3_{2\,2',20}{}^{Z^2_{2\,2',13}} \cdot \tilde{Z}_{1\,2'} \cdot \tilde{Z}_{1'\,2}.$$

The figures which correspond to the braids outside G and F are 8-16. The ones which correspond to the factors in G and F are 17-23 and 24-28 respectively.
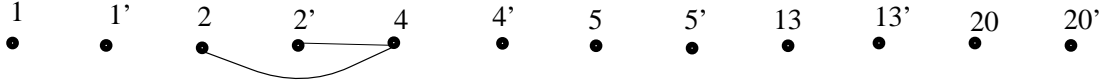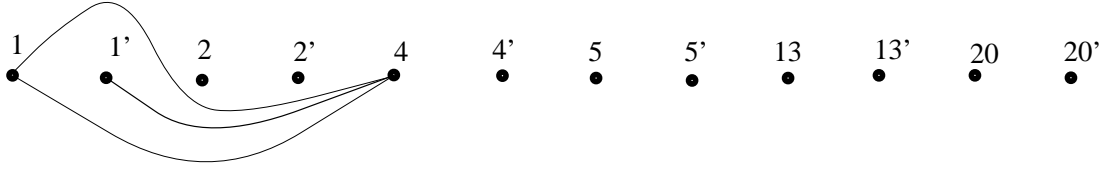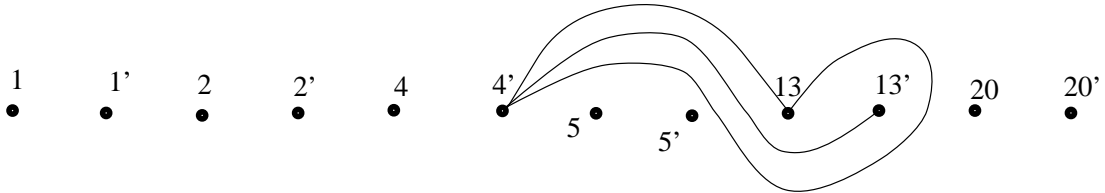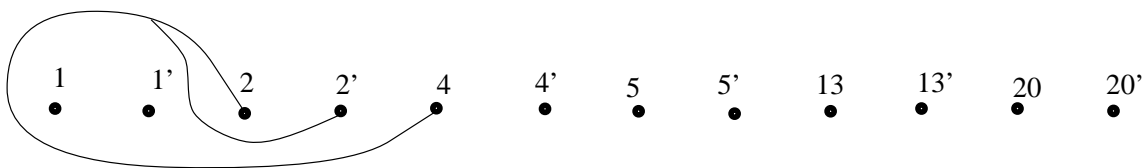

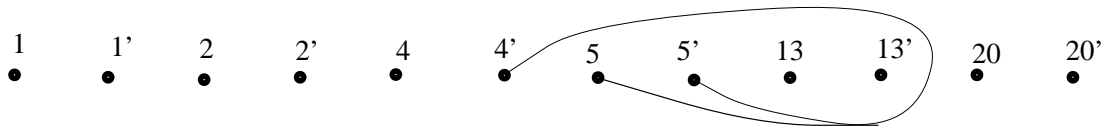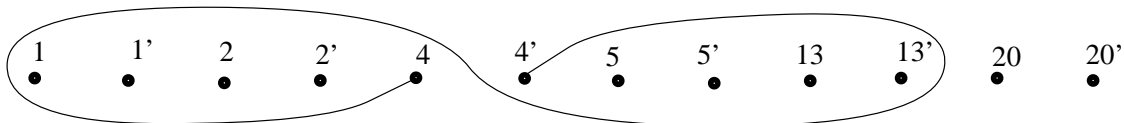FIGURE 8.
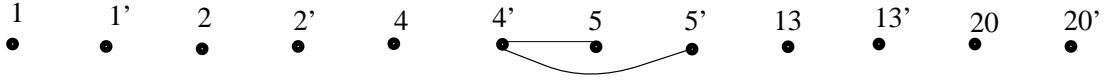
FIGURE 9.

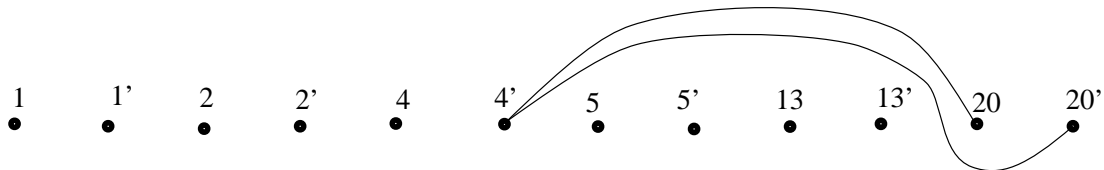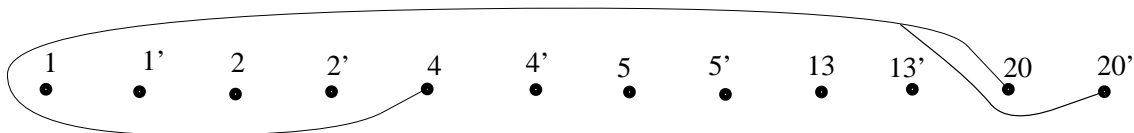FIGURE 10.

FIGURE 11.

FIGURE 12.
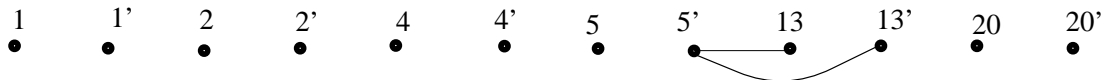

FIGURE 13.
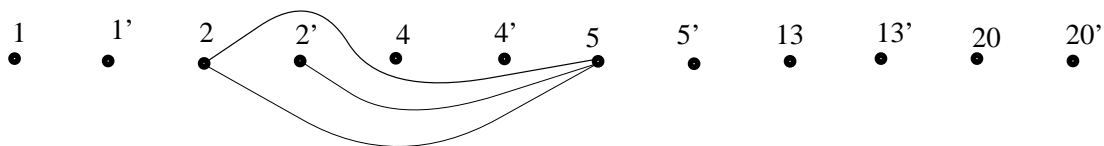

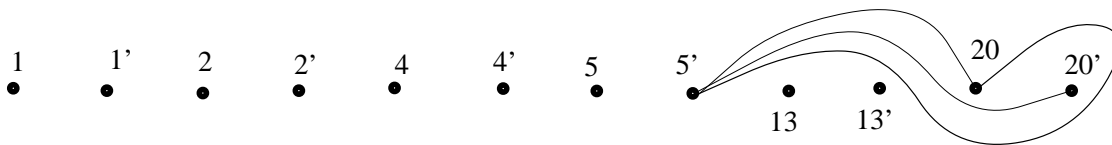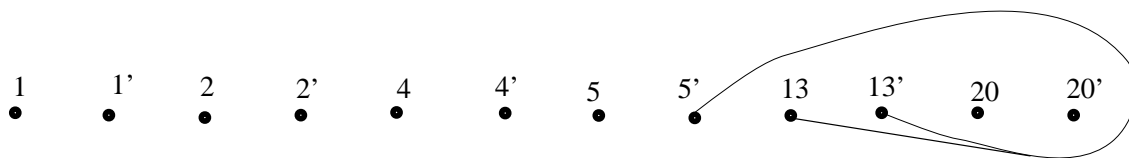FIGURE 14.

FIGURE 15.

FIGURE 16.
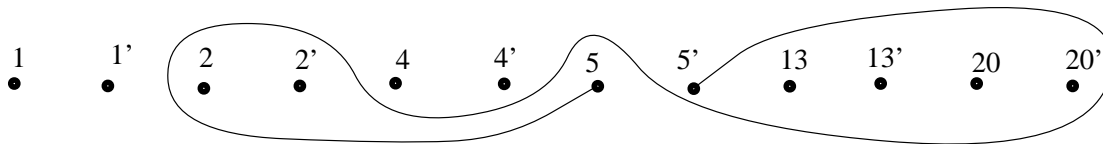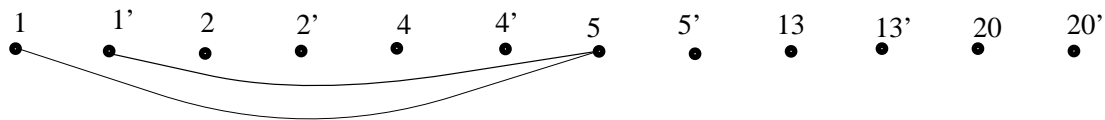

FIGURE 17.

FIGURE 18.
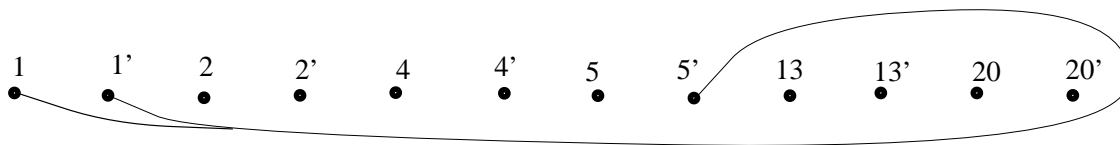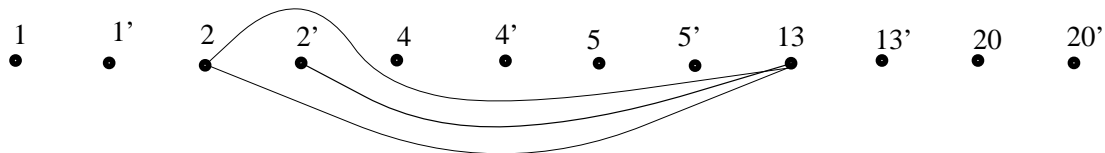

FIGURE 19.

FIGURE 20.

FIGURE 21.

FIGURE 22.

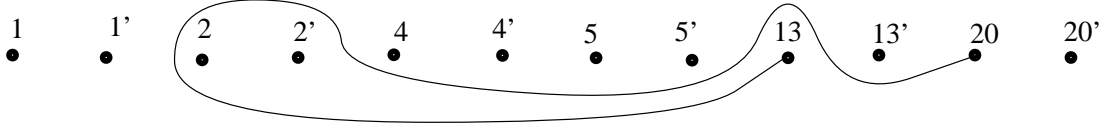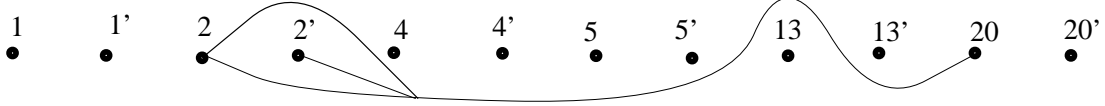FIGURE 23.

FIGURE 24.

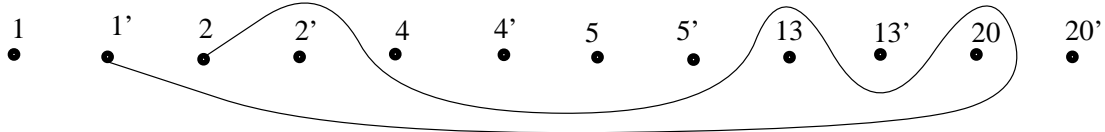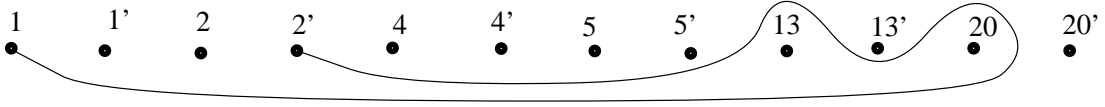FIGURE 25.

FIGURE 26.

FIGURE 27.

FIGURE 28.

In the following theorem we give the local braid monodromies $\varphi_4, \varphi_5, \varphi_7, \varphi_9$ and φ_{10}. Related figures can be easily constructed, using Example 6. We note that φ_{10} was computed precisely in [3] and [18] (and related figures can be found there).

Theorem 14.   (1) *The local braid monodromy φ_4 has the following form*

$$\varphi_4 = Z^2_{4',7\,7'} \cdot Z^{2\,Z^2_{4',7\,7'}}_{4',8\,8'} \cdot Z^3_{3\,3',4} \cdot Z^{3\,Z^2_{4',8\,8'}Z^2_{4',7\,7'}}_{4',15\,15'} \cdot Z^{2\,Z^2_{4',7\,7'}Z^{-2}_{8\,8',15\,15'}}_{4',8\,8'} \cdot$$

$$Z^{2\,Z^{-2}_{7\,7',8\,8'}Z^{-2}_{7\,7',15\,15'}}_{4',7\,7'} \cdot Z_{4\,4'}^{Z^2_{3\,3',4}Z^2_{4',15\,15'}Z^2_{4',8\,8'}Z^2_{4',7\,7'}} \cdot Z^{2\,Z^2_{4,15\,15'}Z^2_{4,8\,8'}Z^2_{4,7\,7'}Z^2_{4\,4'}Z^2_{3\,3',4}}_{4,21\,21'} \cdot$$

$$Z^{2\,Z^2_{4',15\,15'}Z^2_{4',8\,8'}Z^2_{4',7\,7'}}_{4',21\,21'} \cdot \left(G \cdot \left(F \cdot F^{Z^{-1}_{7\,7'}Z^{-1}_{3\,3'}} \right)^{Z^2_{7\,7',8}Z^2_{3\,3',8}} \right)^{Z^2_{3\,3',4}},$$

where

$$G = Z^2_{8',15\,15'} \cdot Z^3_{7\,7',8} \cdot Z^{3\,Z^2_{8',15\,15'}}_{8',21\,21'} \cdot Z^{2\,Z^{-2}_{15\,15',21\,21'}}_{8',15\,15'} \cdot Z_{8\,8'}^{Z^2_{7\,7',8}Z^2_{8',21\,21'}Z^2_{8',15\,15'}} \cdot$$

$$Z^2_{3\,3',8} \cdot Z^{2\,Z^2_{8',21\,21'}Z^2_{8',15\,15'}}_{3\,3',8'}$$

and

$$F = Z^2_{3'\,7} \cdot Z^3_{7',15\,15'} \cdot Z^{2\,Z^2_{7',15\,15'}Z^2_{3'\,7}}_{3'\,7'} \cdot Z^{3\,Z^2_{3'\,7}}_{3',15\,15'} \cdot \tilde{Z}_{15\,21'} \cdot \tilde{Z}_{15'\,21}.$$

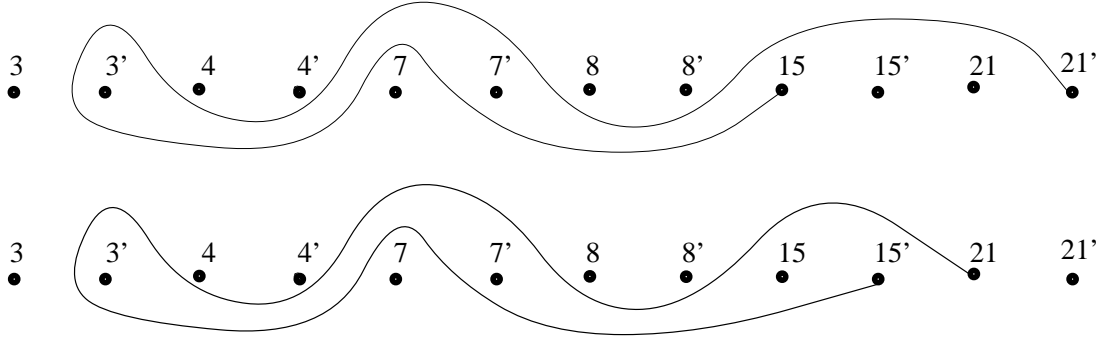*The paths related to $\tilde{Z}_{15\,21'}$ and $\tilde{Z}_{15'\,21}$ appear in Figure 29.*

FIGURE 29.

(2) *The local braid monodromy φ_5 has the following form*

$$\varphi_5 = Z^3_{5',6\,6'} \cdot G^{Z^2_{5',6\,6'}} \cdot Z^3_{5',22\,22'}{}^{Z^2_{5',6\,6'}} \cdot Z_{5\,5'}{}^{Z^2_{5',22\,22'}Z^2_{5',6\,6'}} \cdot Z^2_{5',16\,16'}{}^{Z^2_{5\,5'}Z^2_{5',6\,6'}} \cdot$$

$$Z^2_{5',16\,16'}{}^{Z^2_{5',6\,6'}} \cdot Z^2_{5,11\,11'}{}^{Z^2_{5\,5'}Z^2_{5',6\,6'}} \cdot Z^2_{5',11\,11'}{}^{Z^2_{5',6\,6'}} \cdot Z^2_{5,9\,9'}{}^{Z^2_{5\,5'}Z^2_{5',6\,6'}} \cdot Z^2_{5',9\,9'}{}^{Z^2_{5',6\,6'}},$$

where

$$G = Z^2_{6\,6',9} \cdot Z^3_{9',11\,11'} \cdot Z^2_{6\,6',9'}{}^{Z^2_{6\,6',9}Z^2_{9',11\,11'}} \cdot \left(F \cdot F^{Z^{-1}_{11\,11'}Z^{-1}_{6\,6'}} \right)^{Z^2_{6\,6',9\,9'}Z^2_{9',11\,11'}} \cdot$$

$$Z^2_{9',22\,22'}{}^{Z^2_{9',11\,11'}} \cdot Z^3_{9,22\,22'} \cdot Z^3_{9',16\,16'}{}^{Z^2_{9',11\,11'}} \cdot Z_{9\,9'}{}^{Z^2_{9',16\,16'}Z^2_{9',11\,11'}}$$

and

$$F = Z^2_{6'\,11} \cdot Z^3_{11',16\,16'} \cdot Z^2_{6'\,11'}{}^{Z^2_{11',16\,16'}Z^2_{6'\,11}} \cdot Z^3_{6',16\,16'}{}^{Z^2_{6'\,11}} \cdot \tilde{Z}_{16\,22'} \cdot \tilde{Z}_{16'\,22}.$$

The paths related to $\tilde{Z}_{16\,22'}$ and $\tilde{Z}_{16'\,22}$ are the same ones as in Figure 29, with the following exchange of indices: $3 \mapsto 6, 7 \mapsto 11, 15 \mapsto 16, 21 \mapsto 22$.

(3) *The local braid monodromy φ_7 has the following form*

$$\varphi_7 = Z^2_{8\,8',9} \cdot Z^2_{9',10\,10'} \cdot Z^3_{9',12\,12'}{}^{Z^2_{9',10\,10'}} \cdot Z^2_{9',10\,10'}{}^{Z^{-2}_{10\,10',12\,12'}} \cdot Z^2_{8\,8',9'}{}^{Z^2_{9',12\,12'}Z^2_{9',10\,10'}Z^2_{8\,8',9}} \cdot$$

$$G^{Z^2_{8\,8',9\,9'}Z^2_{9',12\,12'}Z^2_{9',10\,10'}} \cdot Z^2_{9',23\,23'}{}^{Z^2_{9',12\,12'}Z^2_{9',10\,10'}} \cdot Z^3_{9',18\,18'}{}^{Z^2_{9',12\,12'}Z^2_{9',10\,10'}} \cdot Z^2_{9,23\,23'} \cdot$$

$$Z_{9\,9'}{}^{Z^2_{9',18\,18'}Z^2_{9',12\,12'}Z^2_{9',10\,10'}},$$

where

$$G = Z^3_{8',10\,10'} \cdot \left(F \cdot F^{Z^{-1}_{23\,23'}Z^{-1}_{18\,18'}} \right)^{Z^2_{8',10\,10'}} \cdot Z^3_{8',23\,23'}{}^{Z^2_{8',10\,10'}} \cdot Z_{8\,8'}{}^{Z^2_{8',23\,23'}Z^2_{8',10\,10'}} \cdot$$

$$Z^2_{8',18\,18'}{}^{Z^2_{8',10\,10'}} \cdot Z^2_{8,18\,18'}{}^{Z^2_{8\,8'}Z^2_{8',10\,10'}} \cdot Z^2_{8',12\,12'}{}^{Z^2_{8',10\,10'}} \cdot Z^2_{8,12\,12'}{}^{Z^2_{8\,8'}Z^2_{8',10\,10'}}$$

and

$$F = Z^3_{12\,12',18} \cdot Z^2_{18'\,23} \cdot Z^2_{18\,23}{}^{Z^2_{12\,12',18}} \cdot Z^3_{12\,12',23}{}^{Z^2_{12\,12',18}} \cdot \tilde{Z}_{10\,12'} \cdot \tilde{Z}_{10'\,12}.$$

Moreover, the paths related to $\tilde{Z}_{10\ 12'}$ and $\tilde{Z}_{10'\ 12}$ are the same ones as in Figure 28, with the following exchange of indices: $1 \mapsto 10, 2 \mapsto 12, 13 \mapsto 18, 20 \mapsto 23$.

(4) The local braid monodromy φ_9 has the following form

$$\varphi_9 = Z^2_{14',15\ 15'} \cdot Z^3_{13\ 13',14} \cdot \bar{Z}^3_{14',16\ 16'} \cdot Z^{2}{}^{Z^{-2}_{15\ 15',16\ 16'}}_{14',15\ 15'} \cdot Z_{14\ 14'}{}^{Z^2_{13\ 13',14}Z^2_{14',16\ 16'}Z^2_{14',15\ 15'}} \cdot$$

$$\left(\bar{Z}^2_{14,17\ 17'}\right)^{Z^2_{13\ 13',14}} \cdot \bar{Z}^2_{14',17\ 17'} \cdot \left(\bar{Z}^2_{14,18\ 18'}\right)^{Z^2_{13\ 13',14}} \cdot \bar{Z}^2_{14',18\ 18'} \cdot$$

$$\left(G \cdot \left(F \cdot F^{Z^{-1}_{18\ 18'}Z^{-1}_{13\ 13'}}\right)^{Z^{-2}_{17,18\ 18'}}\right)^{Z^2_{13\ 13',14}},$$

where

$$G = Z^2_{16\ 16',17} \cdot Z^3_{17',18\ 18'} \cdot Z^3_{15\ 15',17} \cdot Z_{16\ 16',17}{}^{Z^2_{15\ 15',17}} \cdot Z_{17\ 17'}{}^{Z^2_{17',18\ 18'}Z^2_{16\ 16',17}Z^2_{15\ 15',17}} \cdot$$

$$Z^2_{13\ 13',17} \cdot Z_{13\ 13',17'}{}^{Z^2_{17',18\ 18'}}$$

and

$$F = Z^3_{13',15\ 15'} \cdot Z^3_{16\ 16',18} \cdot \tilde{Z}_{15\ 16'} \cdot \tilde{Z}_{15'\ 16} \cdot Z_{13'\ 18}{}^{Z^2_{13',15\ 15'}} \cdot Z^2_{13\ 18}.$$

The paths related to $\tilde{Z}_{15\ 16'}$ and $\tilde{Z}_{15'\ 16}$ appear in Figure 30.
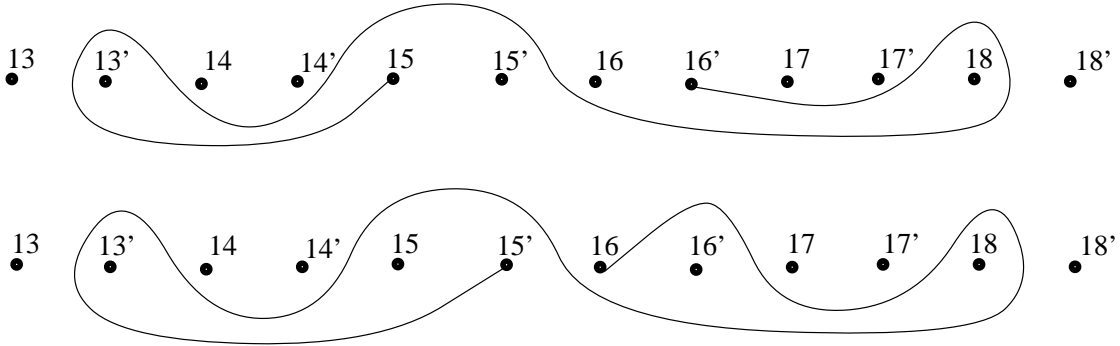
FIGURE 30.

(5) The local braid monodromy φ_{10} has the following form

$$\varphi_{10} = Z^3_{19',20\ 20'} \cdot Z_{24\ 24'}{}^{Z^{-2}_{22\ 22',24}Z^{-2}_{23\ 23',24}} \cdot Z^2_{21\ 21',24'} \cdot \bar{Z}^3_{22\ 22',24} \cdot Z_{21\ 21',24}{}^{Z^{-2}_{23\ 23',24}} \cdot$$

$$G^{Z^2_{19',20\ 20'}} \cdot Z^3_{23\ 23',24} \cdot \bar{Z}^3_{19',21\ 21'} \cdot Z^2_{19,22\ 22'} \cdot Z^2_{19,23\ 23'} \cdot Z^2_{19,24\ 24'},$$

where

$$G = Z^2_{20\ 20',24'} \cdot Z_{20\ 20',24}{}^{Z^{-2}_{23\ 23',24}} \cdot \left(F \cdot F^{Z^{-1}_{23\ 23'}Z^{-1}_{20\ 20'}}\right)^{Z^{-2}_{23\ 23',24}} \cdot Z^2_{19',22\ 22'} \cdot$$

$$Z^2_{19',23\ 23'} \cdot Z^2_{19',24\ 24'} \cdot Z_{19\ 19'}{}^{Z^2_{19',21\ 21'}}$$

and

$$F = Z^3_{20',21\ 21'} \cdot Z^3_{22\ 22',23} \cdot \tilde{Z}_{21\ 22'} \cdot \tilde{Z}_{21'\ 22} \cdot Z_{20'\ 23}{}^{Z^2_{20',21\ 21'}} \cdot Z^2_{20\ 23}.$$

The paths related to $\tilde{Z}_{21\,22'}$ and $\tilde{Z}_{21'\,22}$ are the same ones as in Figure 30, with the following exchange of indices: $13 \mapsto 20, 15 \mapsto 21, 16 \mapsto 22, 18 \mapsto 23$.

5.2.3. *The products C_i.* The deformation also regenerates the parasitic intersection braids $\{\tilde{C}_i\}_{i=1}^{10}$, see Theorem 9. By [18, Lemma 19], we can take the complex conjugations

$$D_1 = D_2 = D_4 = Id \ , \ D_3 = Z^2_{2\,2',3\,3'} \ , \ D_5 = Z^2_{3\,3',5\,5'} \ , \ D_6 = \prod_{i=1,3,4}^{(5)(5')} Z^2_{i\,i',6\,6'} \ ,$$

$$D_7 = \prod_{i=1,2,5,6} Z^2_{i\,i',7\,7'} \ , \ D_8 = \prod_{i=1,2,5,6} Z^2_{i\,i',8\,8'} \ , \ D_9 = \prod_{i=1-4,7}^{(8)(8')} Z^2_{i\,i',9\,9'} \ , \ D_{10} = \prod_{i=1}^{6\ (8)-(9')} Z^2_{i\,i',10\,10'} \ ,$$

$$D_{11} = \prod_{\substack{i=1\\i\neq 5,6,9}}^{10} Z^2_{i\,i',11\,11'} \ , \ D_{12} = \prod_{i=1}^{7\ (11)(11')} Z^2_{i\,i',12\,12'} \ , \ D_{13} = \prod_{\substack{i=3\\i\neq 4,5}}^{12} Z^2_{i\,i',13\,13'} \ ,$$

$$D_{14} = \prod_{\substack{i=1\\i\neq 2,6}}^{12\ (13)(13')} Z^2_{i\,i',14\,14'} \ , \ D_{15} = \prod_{\substack{i=1\\i\neq 3,4,7,8}}^{12\ (13)-(14')} Z^2_{i\,i',15\,15'} \ , \ D_{16} = \prod_{\substack{i=1\\i\neq 5,6,9,11}}^{12\ (13)-(15')} Z^2_{i\,i',16\,16'} \ ,$$

$$D_{17} = \prod_{\substack{i=1\\i\neq 7,10}}^{12\ (13)-(16')} Z^2_{i\,i',17\,17'} \ , \ D_{18} = \prod_{\substack{i=1\\i\neq 8-10}}^{11\ (13)-(17')} Z^2_{i\,i',18\,18'} \ , \ D_{19} = \prod_{\substack{i=2\\i\neq 3}}^{18} Z^2_{i\,i',19\,19'} \ ,$$

$$D_{20} = \prod_{\substack{i=3\\i\neq 4,5,13}}^{18\ (19)(19')} Z^2_{i\,i',20\,20'} \ , \ D_{21} = \prod_{\substack{i=1\\i\neq 3,4,7,8,15}}^{18\ (19)-(20')} Z^2_{i\,i',21\,21'} \ , \ D_{22} = \prod_{\substack{i=1\\i\neq 5,6,9,11,16}}^{18\ (19)-(21')} Z^2_{i\,i',22\,22'} \ ,$$

$$D_{23} = \prod_{\substack{i=1\\i\neq 8-10,12}}^{17\ (19)-(22')} Z^2_{i\,i',23\,23'} \ , \ D_{24} = \prod_{\substack{i=1\\i\neq 11,12}}^{18\ (19)-(23')} Z^2_{i\,i',24\,24'} \ .$$

Let us denote the regenerations of $\tilde{C}_i$ as C_i, $1 \leq i \leq 10$. These are the factorizations of the suitable D_t, as in (1).

5.3. **Properties of the factorization Δ^2_{48}.** The braid monodromy factorization Δ^2_{48} is the product $\prod_{i=1}^{10} C_i\varphi_i$. We now verify that there are no missing braids.

Proposition 15. *The braid monodromy factorization of S is Δ^2_{48}.*

Proof. By Proposition V.2.2 in [14], $\deg\Delta^2_{48} = 48 \cdot 47 = 2256$.

Now we check the degree of the factorization $\prod_{i=1}^{10} C_i\varphi_i$. The monodromies $\{\varphi_i\}_{i=1,3,6,8}$ each consist of six cusps and nine branch points, see Section 5.2.1. We combine their degrees to get $4 \cdot (6 \cdot 3 + 1 \cdot 9) = 4 \cdot 27 = 108$.

In each $\varphi_i, i = 2, 4, 5, 7, 9, 10$ (Section 5.2.2), $\deg F(F)^* = 48$. The factors outside $F(F)^*$ are 20 degree two factors, 12 degree three factors, and 2 degree one factors. Combining

the degrees of these, we get $20 \cdot 2 + 12 \cdot 3 + 2 \cdot 1 = 78$. Thus $\deg \varphi_i = 78 + 48 = 126$ for $i = 2, 4, 5, 7, 9, 10$, and therefore $\deg(\prod_{i=2,4,5,7,9,10} \varphi_i) = 126 \cdot 6 = 756$.

The products $\{C_i\}_{i=1}^{10}$ consist of 696 nodes, see Section 5.2.3. We combine degrees of their factors to get $\deg \prod_{i=1}^{10} C_i = 696 \cdot 2 = 1392$.

Finally $\deg \prod_{i=1}^{10} C_i \varphi_i = 108 + 756 + 1392 = 2256$. Therefore Δ_{48}^2 is the desired braid monodromy factorization. $\square$

We now study invariance properties of Δ_{48}^2. Invariance properties are results in which we prove that the braid monodromy factorization Δ_{48}^2 is invariant under certain elements of B_{48}. Establishing invariance properties is essential in order to simplify the computations which follow from the van Kampen Theorem (see [4]); at the end of this section we describe the effect of these properties on the application of the van Kampen Theorem.

The following definitions are necessary in order to prove the Invariance Properties.

Definition 16. *Let $g_1 \cdots g_k = h_1 \cdots h_k$ be two factorized expressions of the same element in a group G. We say that $g_1 \cdots g_k$ is obtained from $h_1 \cdots h_k$ by a Hurwitz move if $\exists\, 1 \leq p \leq k - 1$, such that $g_i = h_i$ $(i \neq p, p + 1)$ $g_p = h_p h_{p+1} h_p^{-1}$ and $g_{p+1} = h_p$ or $g_p = h_{p+1}$ and $g_{p+1} = h_{p+1}^{-1} h_p h_{p+1}$. In general, $g_1 \cdots g_k \simeq h_1 \cdots h_k$ (Hurwitz equivalent) if $g_1 \cdots g_k$ is obtained from $h_1 \cdots h_k$ by a finite number of Hurwitz moves.*

Definition 17. *Let $g_1 \cdots g_k$ be a factorized expression in G and $h \in G$. We say that $g_1 \cdots g_k$ is invariant under h if it is Hurwitz equivalent to $(g_1)_h \cdots (g_k)_h$, where $(g_i)_h = h^{-1} g_i h$.*

Invariance properties are important in view of Lemma VI.4.2 in [14]: if a braid monodromy factorization Δ_n^2 is invariant under h then the equivalent factorization $(\Delta_n^2)_h$ is also a braid monodromy factorization.

We now quote several invariance rules (the conclusions of Theorems 8, 9, 10), Chakiri's Lemma (see [18]), and make some remarks on invariance properties.

Invariance Rules

(1) A braid Z_{ij}^1 is invariant under $(Z_{ii'} Z_{jj'})^q$ for any $q \in \mathbb{Z}$.
(2) A braid $Z_{i,jj'}^2$ (resp. $Z_{ii',jj'}^2$) is invariant under $Z_{jj'}^q$ (resp. $Z_{ii'}^p Z_{jj'}^q$) for any $p, q \in \mathbb{Z}$.
(3) $Z_{i,jj'}^3$ is invariant under $Z_{jj'}^q$ for any $q \in \mathbb{Z}$.

Lemma 18. (Chakiri). *Let $g = g_1 \cdots g_k$ be a factorized expression in a group G. Then $g_1 \cdots g_k$ is invariant under g^m for any $m \in \mathbb{Z}$.*

Invariance Remarks

(1) To prove invariance of $g_1 \cdots g_k$ under h it is enough to prove that $g_1 \cdots g_t$ and $g_{t+1} \cdots g_k$ are invariant under h. Thus we can divide a factorization into subfactorizations and prove invariance on each part separately.

(2) An element g_1 is invariant under h if and only if g_1 commutes with h.

(3) If a product of elements that commutes with h is invariant under h, the corresponding factorizations are equal.

(4) If two paths σ_1 and σ_2 do not intersect, the corresponding halftwists $H(\sigma_1)$ and $H(\sigma_2)$ commute.

(5) If g is invariant under h_1 and h_2 then g is invariant under $h_1 h_2$.

We finally prove the Invariance Properties of Δ_{48}^2. Denote $Z_{ij} = H(z_{ij})$.

Remark 19. *The halftwists $Z_{ii'}$ and $Z_{jj'}$ commute for all i, j, since the path from i to i' does not intersect the path from j to j'.*

Lemma 20. *Each monodromy among $\varphi_1, \varphi_3, \varphi_6, \varphi_8$ is invariant under $\prod_{j=1}^{24} Z_{jj'}^{m_j}$ for $m_j \in \mathbb{Z}$.*

Proof. The braids in $\varphi_1, \varphi_3, \varphi_6, \varphi_8$, arise from cusps and branch points (see Theorem 12). By Invariance Rule (3), each braid of the form $Z_{ii',j}^3$ is invariant under $Z_{ii'}^q$. Each of the braids of the form Z_{ij}^1 is invariant under $(Z_{ii'} Z_{jj'})^q$, and in particular, if the braid is of the form $Z_{ii'}^1$, then it is invariant under $Z_{ii'}^q$.

Moreover, by Remark 19, each $Z_{kk'}$ commutes with the above braids when $k \neq i, j$.

Therefore, each monodromy φ_i, $i = 1, 3, 6, 8$, is invariant under $\prod_{j=1}^{24} Z_{jj'}^{m_j}$ for $m_j \in \mathbb{Z}$. $\square$

Lemma 21. *Each factorization $\{C_i\}$, $i = 1, \ldots, 10$, is invariant under $\prod_{j=1}^{24} Z_{jj'}^{m_j}$ for $m_j \in \mathbb{Z}$.*

Proof. We apply Invariance Rule (2) and Invariance Remark (4) to each C_i to get the desired invariance. $\square$

We now prove invariance for the monodromy φ_2. A similar proof will apply for each one of the monodromies $\varphi_4, \varphi_5, \varphi_7, \varphi_9, \varphi_{10}$.

Lemma 22. *φ_2 is invariant under $(Z_{4\,4'} Z_{5'\,5'})^p (Z_{13\,13'} Z_{20\,20'})^q (Z_{1\,1'} Z_{2\,2'})^r$ for all $p, q, r \in \mathbb{Z}$.*

Proof. <u>Case 1</u>: $p = q = r$.

As proved in [18, Lemma 12], the monodromy φ_2 can be written as $\prod_{j=1,2,4,5,13,20} Z_{jj'}^{-1} \Delta_{12}^2$. By Lemma 18, φ_2 is invariant under $(\prod_{j=1,2,4,5,13,20} Z_{jj'}^{-1} \Delta_{12}^2)^{-p}$. Since Δ_{12}^2 is a central element, φ_2 is invariant under $(Z_{4\,4'} Z_{5\,5'})^p (Z_{13\,13'} Z_{20\,20'})^p (Z_{1\,1'} Z_{2\,2'})^p$.

<u>Case 2</u>: $p = 0$.

Let $\epsilon = (Z_{13\,13'} Z_{20\,20'})^q (Z_{1\,1'} Z_{2\,2'})^r$.

(1) <u>Step 1</u>: Factors outside of $F \cdot F^{Z_{20\ 20'}^{-1} Z_{13\ 13'}^{-1}}$.

$Z_{4\ 4'}^{Z_{4',13\ 13'}^2 Z_{4',5\ 5'}^2 Z_{2\ 2',4}^2 Z_{1\ 1',4}^2}$ and $Z_{5\ 5'}^{Z_{2\ 2',5}^2 Z_{5',20\ 20'}^2 Z_{5',13\ 13'}^2 Z_{2\ 2',4}^2 Z_{1\ 1',4}^2}$ commute with ϵ.

$Z_{1\ 1',4}^3, Z_{4',13\ 13'}^3{}^{Z_{4',5\ 5'}^2}, Z_{2\ 2',5}^3{}^{Z_{2\ 2',4}^2 Z_{1\ 1',4}^2}$ and $Z_{5',20\ 20'}^3{}^{Z_{5',13\ 13'}^2 Z_{2\ 2',4}^2 Z_{1\ 1',4}^2}$ are invariant under ϵ by Invariance Rule (3).

The degree 2 factors are of the form $Z_{\alpha\alpha',\beta}^2$ where $\beta = 4, 4', 5, 5'$. By Invariance Rule (2), they are invariant under $Z_{\alpha\alpha'}$, and since the other halftwists in ϵ commute with $Z_{\alpha\beta}$ and $Z_{\alpha'\beta}$, we get that $Z_{\alpha\alpha',\beta}^2$ is invariant under ϵ.

Moreover all conjugations which appear in φ_2 (i.e. $(\ )^*$) are invariant under ϵ by Invariance Rule (2) and by Invariance Remark (4).

(2) <u>Step 2</u>: Factors in $F \cdot F^{Z_{20\ 20'}^{-1} Z_{13\ 13'}^{-1}}$.

Let $\rho = Z_{13\ 13'} Z_{20\ 20'}$. In order to prove that $F \cdot F^{\rho^{-1}}$ is invariant under ϵ, we consider the following subcases:

- <u>Subcase 2.1</u>: $q = 0$ and $\epsilon = (Z_{1\ 1'} Z_{2\ 2'})^r$.

  $Z_{2\ 2',13}^3$ and $Z_{2\ 2',20}^3{}^{Z_{2\ 2',13}^2}$ are invariant under $Z_{2\ 2'}$ (Invariance Rule (3)) and commute with $Z_{1\ 1'}$ (Invariance Remark (4)). Thus $Z_{2\ 2',13}^3 Z_{2\ 2',20}^3{}^{Z_{2\ 2',13}^2}$ is invariant under ϵ (Invariance Remarks (1) and (5)). $Z_{13'\ 20}^2$ and $\cdot (Z_{13\ 20}^2)^{Z_{2\ 2',13}^2}$ commute with $Z_{1\ 1'}$ and $Z_{2\ 2'}$ and thus with ϵ. $\tilde{Z}_{1\ 2'} \cdot \tilde{Z}_{1'\ 2}$ is invariant under $(Z_{1\ 1'} Z_{2\ 2'})^r$ by Invariance Rule (1). For $F^{\rho^{-1}}$ we have the same conclusion, since $Z_{20\ 20'}^{-1} Z_{13\ 13'}^{-1}$ commutes with $\epsilon = (Z_{1\ 1'} Z_{2\ 2'})^r$.

- <u>Subcase 2.2</u>: $r = 0$, $q = 1$ and $\epsilon = Z_{13\ 13'} Z_{20\ 20'}$.

  Note that in this case $\epsilon = \rho$. To prove that $F \cdot F^{\rho^{-1}}$ is invariant under ρ, we must show that $F \cdot F^{\rho^{-1}}$ is Hurwitz equivalent to $F^\rho \cdot F$. Since AB is Hurwitz equivalent to BA^B, it is enough to prove that $F \cdot F^{\rho^{-1}}$ is Hurwitz equivalent to $F \cdot (F^\rho)^F$. Thus it is enough to prove that $F^{\rho^{-1}}$ is Hurwitz equivalent to $(F^\rho)^F$ or that $(F^{\rho^{-1}})^{F^{-1}}$ is Hurwitz equivalent to F^ρ.

  By Theorem 13, $F = \Delta_8^2 \rho^{-2} Z_{1\ 1'}^{-2} Z_{2\ 2'}^{-2} (F^{-1})^{\rho^{-1}}$, thus $F^{-1} = F^{\rho^{-1}} (Z_{1\ 1'} Z_{2\ 2'})^2 \rho^2 \Delta_8^2$. Now we have

  $$(F^{\rho^{-1}})^{F^{-1}} = (F^{\rho^{-1}})^{F^{\rho^{-1}} (Z_{1\ 1'} Z_{2\ 2'})^2 \rho^2 \Delta_8^2} \underset{\substack{as\ factorized \\ expression}}{\overline{\overline{=}}} (F^{\rho^{-1}})^{F^{\rho^{-1}} (Z_{1\ 1'} Z_{2\ 2'})^2 \rho^2}$$

  $$\underset{Chakiri}{\widetilde{=}} (F^{\rho^{-1}})^{(Z_{1\ 1'} Z_{2\ 2'})^2 \rho^2} = (F^{(Z_{1\ 1'} Z_{2\ 2'})^2})^\rho \underset{Subcase\ 2.1}{\widetilde{=}} F^\rho.$$

- <u>Subcase 2.3</u>: $q = 2q'$; $\epsilon = (Z_{13\ 13'} Z_{20\ 20'})^{2q'} (Z_{1\ 1'} Z_{2\ 2'})^r$.

  $F \cdot F^{\rho^{-1}}$ can be written as $\Delta_8^2 (Z_{13\ 13'} Z_{20\ 20'})^{-2} (Z_{1\ 1'} Z_{2\ 2'})^{-2}$. By Chakiri's Lemma, $F \cdot F^{\rho^{-1}}$ is invariant under $(\Delta_8^2 (Z_{13\ 13'} Z_{20\ 20'})^{-2} (Z_{1\ 1'} Z_{2\ 2'})^{-2})^{-q'}$ and thus under $(Z_{13\ 13'} Z_{20\ 20'})^{2q'} (Z_{1\ 1'} Z_{2\ 2'})^{2q'}$. By Subcase 2.1, $F \cdot F^{\rho^{-1}}$ is invariant under

$(Z_{1\,1'}Z_{2\,2'})^{r-2q'}$. By Invariance Remark (5), $F \cdot F^{\rho^{-1}}$ is invariant under

$$(Z_{13\,13'}Z_{20\,20'})^{2q'}(Z_{1\,1'}Z_{2\,2'})^{2q'}(Z_{1\,1'}Z_{2\,2'})^{r-2q'} = \epsilon.$$

- <u>Subcase 2.4</u>: $q = 2q' + 1$; $\epsilon = (Z_{13\,13'}Z_{20\,20'})^{2q'+1}(Z_{1\,1'}Z_{2\,2'})^r$.

  It is easy to verify this case by using the prior cases 2.2, 2.3 and Invariance Remark (5).

  <u>Case 3</u>: p, q, r arbitrary; $\epsilon = (Z_{4\,4'}Z_{5\,5'})^p(Z_{13\,13'}Z_{20\,20'})^q(Z_{1\,1'}Z_{2\,2'})^r$.

By case 1, φ_2 is invariant under $(Z_{4\,4'}Z_{5\,5'})^p(Z_{13\,13'}Z_{20\,20'})^p(Z_{1\,1'}Z_{2\,2'})^p$, and by case 2 under $(Z_{13\,13'}Z_{20\,20'})^{q-p}(Z_{1\,1'}Z_{2\,2'})^{r-p}$. By Invariance Remark (5), φ_2 is invariant under ϵ. $\qquad\square$

Corollary 23. *Each one of the monodromies $\varphi_4, \varphi_5, \varphi_7, \varphi_9, \varphi_{10}$ is invariant under $(Z_{ii'}Z_{jj'})^p(Z_{kk'}Z_{ll'})^q(Z_{mm'}Z_{nn'})^r$ for all $p, q, r \in \mathbb{Z}$, where i and j are the diagonal lines around the relevant 6-point, and k and l (resp. m and n) are the vertical (resp. horizontal) ones (see Figure 1).*

Since the proof of the invariance for the 6-points relies on the invariance of each local braid monodromy under such expressions as in Corollary 23, we have to pay attention that each diagonal line (except the lines 14, 17, 19, 24 in Figure 1) at a certain 6-point is also a diagonal line at another 6-point. For example, the monodromy φ_2 is invariant under $(Z_{4\,4'}Z_{5\,5'})^p(Z_{13\,13'}Z_{20\,20'})^q(Z_{1\,1'}Z_{2\,2'})^r$ for all $p, q, r \in \mathbb{Z}$ (see Lemma 22). In addition, the monodromy φ_4 is invariant under $(Z_{4\,4'}Z_{8\,8'})^{p'}(Z_{3\,3'}Z_{7\,7'})^{q'}(Z_{15\,15'}Z_{21\,21'})^{r'}$ for all $p', q', r' \in \mathbb{Z}$ (see Corollary 23). This implies that $p = p'$. In the same way, φ_5 is invariant under $(Z_{5\,5'}Z_{9\,9'})^{p''}(Z_{6\,6'}Z_{11\,11'})^{q''}(Z_{16\,16'}Z_{22\,22'})^{r''}$ for all $p'', q'', r'' \in \mathbb{Z}$. This implies that $p = p''$. In the notation of Lemmas 20 and 21, $p = m_4 = m_5, p' = m_4 = m_8, p'' = m_5 = m_9$, and therefore $m_4 = m_5 = m_8 = m_9$. Now, the lines 14 and 17 (resp. 19 and 24) are diagonal lines at only one 6-point 9 (resp. 10). Therefore in the above notation, $m_{14} = m_{17}$ (resp. $m_{19} = m_{24}$).

We now consider the vertical lines. By a similar argument as above, we can conclude that $m_{13} = m_{18} = m_{20} = m_{23}$ from the invariance of $\varphi_2, \varphi_7, \varphi_9$ and φ_{10}. Since the lines 6 and 11 (resp. 3 and 7) are vertical lines at only one 6-point 5 (resp. 4), we get $m_6 = m_{11}$ (resp. $m_3 = m_7$).

For the horizontal lines, we can immediately conclude that $m_1 = m_2, m_{10} = m_{12}$, and $m_{15} = m_{16} = m_{21} = m_{22}$.

Corollary 24. *The braid monodromy factorization Δ_{48}^2 is invariant under*

$$\rho = (Z_{1\,1'}Z_{2\,2'})^{m_1}(Z_{6\,6'}Z_{11\,11'})^{m_6}(Z_{10\,10'}Z_{12\,12'})^{m_{10}}(Z_{3\,3'}Z_{7\,7'})^{m_3}(Z_{14\,14'}Z_{17\,17'})^{m_{14}}(Z_{19\,19'}Z_{24\,24'})^{m_{19}}$$

$$(Z_{4\,4'}Z_{5\,5'}Z_{8\,8'}Z_{9\,9'})^{m_4}(Z_{13\,13'}Z_{18\,18'}Z_{20\,20'}Z_{23\,23'})^{m_{13}}(Z_{15\,15'}Z_{16\,16'}Z_{21\,21'}Z_{22\,22'})^{m_{15}}.$$

5.4. **Consequences from the Invariance Theorems.** We are interested in the fundamental group $\pi_1(\mathbb{CP}^2 - S)$ and the fundamental group of the Galois cover of the surface (with respect to the generic projection onto $\mathbb{CP}^2$). These groups are computed in [4].

Here we extract qualitative information concerning the importance of the above computations and how they are connected to the groups.

The van Kampen Theorem [20] induces a finite presentation of the fundamental group of complements of curves by means of generators and relations. This is done by applying the theorem on the resulting factorization Δ_{48}^2. We take any path from k to ℓ, cut it in M, then towards k along the path, around k and coming back the same way. Consider A as an element of the fundamental group. Do the same to ℓ to obtain B. See Figure 31. Now, according to the van Kampen Theorem we obtain a relation which involves A and B. The
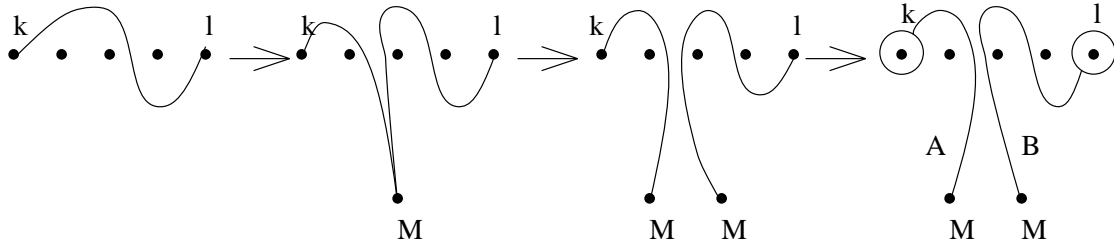

FIGURE 31.

following example is illustrative. Consider Figure 32. We construct B by proceeding from M towards $5'$ above $6'$ and 6 encircling $5'$ counterclockwise and proceeding back above 6 and $6'$. Therefore $B = 6'65'6^{-1}6'^{-1}$. In a similar way, $A = 52'21'151^{-1}1'^{-1}2^{-1}2'^{-1}5^{-1}$. If this path is obtained from a branch point (by Moishezon-Teicher monodromy), then by the van Kampen Theorem, we have the relation $6'65'6^{-1}6'^{-1} = 52'21'151^{-1}1'^{-1}2^{-1}2'^{-1}5^{-1}$.
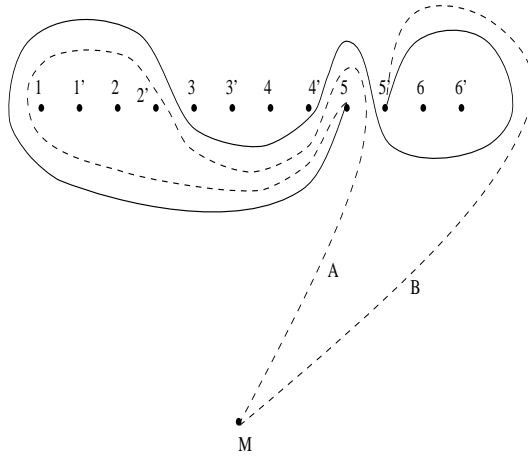

FIGURE 32.

In a similar way, consider paths which are related to cusps (Figure 33) and nodes (Figure 34). From Figure 33 we get three relations $BAB = ABA$, $BA'B = A'BA'$ and $BA''B = A''BA''$. And from Figure 34, we get $AB = BA$ and $A'B = BA'$.
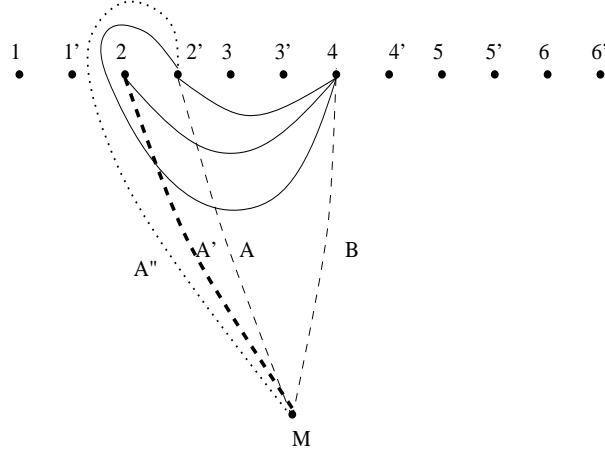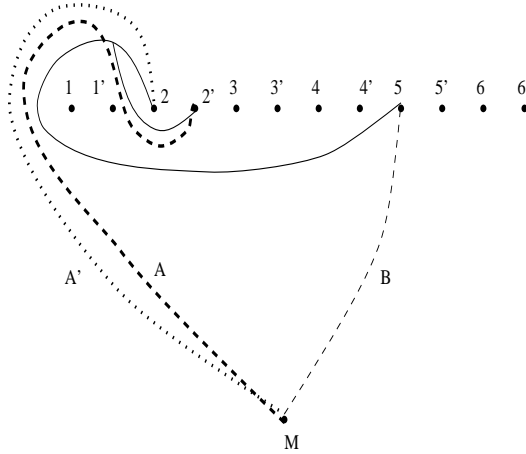

FIGURE 33.

FIGURE 34.

We now explain the importance of the invariance of Δ^2_{48} (Corollary 24). Let z_{ij} be a path connecting i or i' with j or j' and Z_{ij} its corresponding halftwist. We can conjugate Z_{ij} by $Z^{\pm 1}_{ii'}, Z^{\pm 1}_{jj'}$. These conjugations are the actions of $Z^{\pm 1}_{ii'}$ (resp. $Z^{\pm 1}_{jj'}$) on the "head" (resp. "tail") of z_{ij} within a small circle around i and i' (resp. j and j'). The "body" of z_{ij} does not change under such conjugations and in particular not under $Z^{\pm m}_{ii'}$ or $Z^{\pm n}_{jj'}$ for $m, n \in \mathbb{Z}$. Therefore, Corollary 24 enables us to use Theorem 1.6 from [19]:

Theorem 25. *If a sub-factorization $\prod_{i=s}^{r} Z_i$ is invariant under any element h, and $\prod_{i=s}^{r} Z_i$ induces a relation $A_{i_1} \cdot ... \cdot A_{i_t}$ on $\pi_1(\mathbb{CP}^2 - S)$ via the van Kampen Theorem, then $(A_{i_1})_h \cdot ... \cdot (A_{i_t})_h$ is also a relation.*

That means that we can expand our list of relations, which assists us in the computations in [4]:

Corollary 26. *Consider* $\rho = \prod\limits_{j=1}^{24} \rho_{m_j} = \prod\limits_{j=1}^{24} Z_{jj'}^{m_j}$ *from Corollary 24. If R is any relation in* $\pi_1(\mathbb{CP}^2 - S)$, *then R_ρ is also a relation in* $\pi_1(\mathbb{CP}^2 - S)$, *where R_ρ is the relation induced from R by replacing A_j with $(A_j)_{\rho_{m_j}}$.*

6. Acknowledgements

References

[1] Artin, E., Theorie der Zoepfe, Hamburg Abh. **4**, 1926, 47-72.

[2] Artin, E., *Theory of braids*, The annals of Mathematics **48**, 1947, 101-126.

[3] Amram, M., *Galois Covers of Algebraic Surfaces,* Ph.D. Dissertation, 2001.

[4] Amram, M., Ciliberto, C., Miranda, R., Teicher, M., Vishne, U., *Fundamental group of a Galois cover of a non-prime $K3$ surface* , preprint.

[5] Amram, M., Teicher, M., *On the degeneration, regeneration and braid monodromy of $T \times T$*, Acta Applicandae mathematicae, 75(1), 2003, 195-270.

[6] Amram, M., Teicher, M., Vishne, U., *The fundamental group of the Galois cover for Hirzebruch surface $F_1(2, 2)$*, International J. of Algebra and Computation, to appear.

[7] Auroux, D., Donaldson, S., Katzarkov, L., Yotov., M., *Fundamental groups of complements of plane curves and symplectic invariants*, Topology **43**, 2004, 1285-1318.

[8] Chisini, O., Courbes de diramation des planes multiple et tresses algebriques, Deuxieme Colloque de Geometrie Algebrique tenu a Liege, CBRM, 1952, 11-27.

[9] Ciliberto, C., Lopez, A., Miranda, R., *Projective Degenerations of K3 Surfaces, Gaussian Maps, and Fano Threefolds.* Inventiones mathematicae, vol. 114, 1993, 641-667.

[10] Ciliberto, C., Miranda, R., Teicher, M., *Pillow degenerations of K3 surfaces*, Applications of Algebraic Geometry to Coding Theory, Physics and Computations, NATO Science Series II/36, Kluwer Acad. Publish., 2001, 53-64.

[11] Kulikov, V., *Degenerations of K3 Surfaces and Enriques Surfaces.* Math. USSR Izvestija 11, 1977, 957-989.

[12] Moishezon, B., *Algebraic surfaces and the arithmetic of braids, I*, Arithmetic and Geometry, papers dedicated to I.R. Shafarevich, Birkhäuser, 1983, 199-269.

[13] Moishezon, B., Teicher, M., *Simply connected algebraic surfaces of positive index*, Invent. Math. **89**, 1987, 601-643.

[14] Moishezon, B., Teicher, M., *Braid group technique in complex geometry I, Line arrangements in* $\mathbb{CP}^2$, Contemporary Math. **78**, 1988, 425-555.

[15] Moishezon, B., Teicher, M., *Braid group technique in complex geometry II, From arrangements of lines and conics to cuspidal curves*, Algebraic Geometry, Lect. Notes in Math. Vol. **1479**, 1990.

[16] Moishezon, B., Teicher, M., *Finite fundamental groups, free over* $\mathbb{Z}/c\mathbb{Z}$, *Galois covers of* $\mathbb{CP}^2$, Math. Ann. **293**, 1992, 749-766.

[17] Moishezon, B., Teicher, M., *Braid group technique in complex geometry III: Projective degeneration of* V_3, Contemp. Math. **162**, 1993, 313-332.

[18] Moishezon, B., Teicher, M., *Braid group technique in complex geometry IV: Braid monodromy of the branch curve* S_3 *of* $V_3 \to \mathbb{CP}^2$ *and application to* $\pi_1(\mathbb{CP}^2 - S_3, *)$, Contemporary Math. **162**, 1993, 332-358.

[19] Moishezon, B., Teicher, M., *Fundamental groups of complements of branch curves as solvable groups*, Israel Mathematics Conference Proceedings (AMS Publications), vol. 9, 1996, 329-346.

[20] van Kampen, E.R., *On the fundamental group of an algebraic curve*, Amer. J. Math. **55**, 1933, 255-260.

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3

2

4

1

9 10

13

18

16

1

6

5

1

i 12 i' j

j' 7

8

A B

11

i 3 i' M 19 A= Γ_i

j j'

D= Γ_j

$19'$

i i' 4 j

j' 1

-3

C D 2

M C= $\Gamma_{i'}$ Γ_i $\Gamma_{i'}^{-1}$

D= $\Gamma_{j'}$